MATH2050a Mathematical Analysis I

Exercise 3 suggested Solution

7. If $\{b_n\}$is a bounded sequence and $\lim a_n = 0$. Show that $\lim a_n b_n = 0$. Explain why Theorem 3.2.3 cannot be used.

Solution:

Suppose that the sequence $\{b_n\}$is bounded, and $\lim a_n = 0$. Now let $|b_n| < M$, for some $M > 0$. And for each $\epsilon > 0$, there exists $n_\epsilon \in N$, $\forall n > n_\epsilon$, $|a_n - 0| < \frac{1}{M\epsilon}$. So, when $n > n_\epsilon$, we have

$|a_n b_n - 0| < M|a_n| < M\frac{1}{M\epsilon} = \epsilon$ Therefore, $\lim a_n b_n = 0$.

20. Let $\{x_n\}$be a sequence of positive real numbers such that $\lim x_n^{1/n} = L$. Show that there exists a number r with $0 < r < 1$ such that $0 < x_n < r^n$ for all sufficiently large $n \in N$. Use this to show that $\lim x_n = 0$.

Solution:

Since $L < 1$, there exists $\epsilon_0 > 0$, such that $L + \epsilon_0 < 1$.Put $r = L + \epsilon_0$, then $0 < r < 1$.

Since $\lim x_n^{1/n} = L$, for the above $\epsilon_0 > 0$, for all sufficiently large $n \in N$, we have

$$x_n^{1/n} < L + \epsilon_0$$

Hence, $x_n^{1/n} < r$, which implies that $0 \leq x_n < r^n$. Since $0 < r < 1$, $\lim r^n = 0$, So $\lim x_n = 0$.

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in N$. Show that $\{x_n\}$ is bounded and monotone. Find the limit.

Solution:

Since $x_{n+1} = \frac{1}{2}x_n + 2$, change the form, we have $x_{n+1} - 4 = \frac{1}{2} \times (x_n - 4)$.Define

$y_n = x_n - 4$, then

$$y_1 := 4 \text{ and } y_{n+1} = \frac{1}{2}y_n$$

Hence, the sequence $\{y_n\}$ is a geometric sequence. $y_n = y_1(\frac{1}{2})^{n-1} = 4(\frac{1}{2})^{n-1}$. So $x_n - 4 = 4(\frac{1}{2})^{n-1}$. Obviously, $\forall n \in N$, $0 < x_n - 4 < 4$, $\{x_n\}$ is bounded. The monotonicity of $\{x_n\}$ is obtained from the expression : $x_n = 4 + 4(\frac{1}{2})^{n-1}$. Since $\lim(\frac{1}{2})^{n-1} = 0$, we have $\lim x_n = 4$.

11.Let $x_n := 1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2$ for each $n \in N$. Prove that $\{x_n\}$ is increasing and bounded, and hence converges. [Hint: Note that if k$\geq 2, then 1/$k$^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$.]

Solution:

Since $x_n := 1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2$, by replacing n=n+1, we obtain $x_{n+1} := 1/1^2+1/2^2+1/3^2+...+1/n^2+1/(n+1)^2$, so $x_{n+1}-x_n = 1/(n+1)^2 > 0$. Therefore, $\{x_n\}$ is increasing.

since $1/k^2 \leq 1/k(k-1) = 1/(k-1) - 1/k$, we have

$$\begin{aligned} x_n &= 1/1^2 + 1/2^2 + 1/3^2 + ... + 1/n^2 \\ &\leq 1 + 1/1 \times 2 + 1/2 \times 3 + ... + 1/(n-1) \times n \\ &= 1 + 1 - 1/2 + 1/2 - 1/3 + ... + 1/(n-1) - 1/n \\ &= 1 + 1 - 1/n \\ &< 2 \end{aligned}$$

Since $0 < x_n < 2$, $\{x_n\}$ is bounded. If a sequence monotone and bounded, then it converges. Therefore, $\{x_n\}$ converges.